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Recursion operators for infinitesimal transformations and their inverses for certain nonlinear evolution equations

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Abstract. Recursion operators generating infinitesimal transformations have been obtained systematically for certain nonlinear evolution equations. Their inverses have been evaluated and a double infinity of infinitesimal transformations and nonlinear equations generated.

1. Introduction

Recently, Sasaki and Bullough (1981) have derived a double infinity of conservation laws for the sine-Gordon (SG) equation. One set of these conserved quantities, which are non-local, leads to a hierarchy of SG equations while the other leads to nonlinear evolution equations (NLEE) related to the modified KdV (MKdV) equations. Wadati (1978) has shown that about every solution $u(x, t)$ of the KdV equation there exist a single infinity of infinitesimal transformations (IT), that is, functions $y(x, t)$ such that $u(x, t)+\varepsilon y(x, t)$ are also solutions of the KdV equation to terms linear in ε. He has further established a one–one correspondence between the IT and the single infinity of conservation laws. This suggests a double infinity of IT about any solution of the SG equation.

In this paper we establish the existence of a double infinity of IT about every solution of the SG and hence MKdV equation. These IT are obtained through a recursion operator and its inverse, each of which generates an infinity of IT. The recursion operator and its inverse also generate a two-fold infinite hierarchy of NLEE which agree with those obtained by Sasaki and Bullough (1981). This method can be extended to the KdV equation. We obtain the inverse of the recursion operator for the IT of the KdV equation, using the Miura transformation connecting the KdV and MKdV equations. This inverse generates an infinity of IT, other than those obtained by Wadati (1978). It also leads to a new hierarchy of NLEE.

We first find the recursion operators for the IT of the nonlinear Schrödinger (NLS), SG, MKdV and KdV equations using certain naturally occurring recursion relations in the generalised Wronskian techniques applied to the Zakharov–Shabat–AKNS system of equations (Calogero and Degasperis 1976a)

$$\psi_{1x}+\mathrm{i}\zeta\psi_1=q(x, t)\psi_2, \qquad \psi_{2x}-\mathrm{i}\zeta\psi_2=r(x, t)\psi_1. \tag{1.1}$$

This method of finding the recursion operator for the IT has not been reported before and might prove to be a systematic method for finding the recursion operator for NLEE whose spectral problem is a system of $n\times n$ equations with $n\geqslant 2$. This is detailed in § 2. It will also turn out that the recursion operator depends only on the relation between $q(x,t)$ and $r(x,t)$. In § 3 we derive the inverse of the recursion operator for the SG and MKdV equations obtained in § 2. The inverse of the recursion operator for the KdV equation is obtained in § 4.

We wish to remark here that the recursion operator and its inverse, in the case of the KdV equation, are singular, that is, there exist non-zero functions which are mapped to zero by these operators. Thus the inverses are generalised inverses. In the finite-dimensional case it is known that generalised inverses of singular matrices are not unique. Whether there exist other inverses for these singular operators has not been investigated in this paper.

2. Derivation of the recursion operator

Following Calogero and Degasperis (1976a), we write (1.1) in the form

$$\psi_x + \mathrm{i}k\sigma_3\psi = (q_1\sigma_1 + \mathrm{i}q_2\sigma_2)\psi \tag{2.1}$$

where $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$, T denoting transpose, σ_i, $i = 1, 2, 3$, are the Pauli spin matrices and $q_1(x,t)$, $q_2(x,t)$ are related to $q(x,t)$ and $r(x,t)$ of (1.1) by

$$q_1 = \tfrac{1}{2}(q+r), \qquad q_2 = \tfrac{1}{2}(q-r).$$

The generalised Wronskian relation for (2.1) is

$$\begin{aligned}\psi'^{\mathrm{T}}F\psi\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \mathrm{d}x\, \psi'^{\mathrm{T}}(k,x)(&-\mathrm{i}k\{\sigma_3, F(x)\} + \tfrac{1}{2}S_1(x)\{\sigma_1, F(x)\}\\ &-\tfrac{1}{2}\mathrm{i}S_2(x)[\sigma_2, F(x)] + \tfrac{1}{2}D_1(x)[\sigma_1, F(x)]\\ &-\tfrac{1}{2}\mathrm{i}D_2(x)\{\sigma_2, F(x)\} + F_x(x))\psi(k,x)\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &S_j = q_j' + q_j, \qquad D_j = q_j' - q_j, \qquad j = 1, 2,\\ &[A,B] = AB - BA, \qquad \{A,B\} = AB + BA,\end{aligned} \tag{2.3}$$

$q = (q_1, q_2)^{\mathrm{T}}$ and $q' = (q_1', q_2')^{\mathrm{T}}$ are two potentials and $\psi(k,x)$ and $\psi'(k,x)$ are the corresponding solutions of (2.1). F is an arbitrary 2×2 matrix. The recursion relation which follows from (2.2) is (see equation (3.2.5) of Calogero and Degasperis 1976a)

$$\{\sigma_3, F^{(n+1)}\} = \tfrac{1}{2}(S_1\{\sigma_1, F^{(n)}\} - \mathrm{i}S_2[\sigma_2, F^{(n)}] + D_1[\sigma_1, F^{(n)}] - \mathrm{i}D_2\{\sigma_2, F^{(n)}\}) + F_x^{(n)}. \tag{2.4}$$

With $q = q'$ the recursion relations for the matrix elements of $F^{(n)}$ and $F^{(n+1)}$ are

$$\begin{aligned} &2F_{11}^{(n+1)} = 2rF_{12}^{(n)} + (F_{11}^{(n)})_x, \qquad -2F_{22}^{(n+1)} = 2qF_{12}^{(n)} + (F_{22}^{(n)})_x,\\ &-(F_{12}^{(n)})_x = qF_{11}^{(n)} + rF_{22}^{(n)}.\end{aligned} \tag{2.5}$$

From (2.5) one obtains

$$2F_{11}^{(n+1)} = (F_{11}^{(n)})_x - 2r\int^x g^{(n)}(x_1,t)\,\mathrm{d}x_1, \tag{2.6a}$$

$$2F_{22}^{(n+1)} = -(F_{22}^{(n)})_x + 2q\int^x g^{(n)}(x_1,t)\,\mathrm{d}x_1, \tag{2.6b}$$

where

$$g^{(n)} = qF_{11}^{(n)} + rF_{22}^{(n)}. \tag{2.6c}$$

2.1. Recursion operator for NLSE

We put $r(x, t) = -q^*(x, t)$ corresponding to the NLS equation (* denoting complex conjugate). With $F_{22}^{(n+1)} = y^{(n+1)}$, $F_{22}^{(n)} = -\mathrm{i}y^{(n)}$ and $F_{11}^{(n)} = \mathrm{i}y^{(n)*}$, equation (2.6b) becomes

$$T(q)\{y^{(n)}\} \equiv y^{(n+1)} = \mathrm{i}\,\partial y^{(n)}/\partial x + 2\mathrm{i}q \int^x (qy^{(n)*} + q^* y^{(n)})\,\mathrm{d}x_1 \tag{2.7}$$

which is the recursion relation for the IT $y^{(n)}(x, t)$ about any solution $q(x, t)$ of the NLSE,

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0. \tag{2.8}$$

This is proved by showing that if $y^{(n)}(x, t)$ satisfies

$$y_t - \mathrm{i}y_{xx} - 2\mathrm{i}(2|q|^2 y + q^2 y^*) = 0, \tag{2.9}$$

the evolution equation satisfied by an IT $y(x, t)$ about a solution $q(x, t)$ of (2.8), then $y^{(n+1)}(x, t)$ given by (2.7) also satisfies (2.9).

A hierarchy of NLEE related to the NLSE is then

$$q_t - [T(q)]^m\{q_x\} = 0. \tag{2.10}$$

For $m = 1$ one gets (2.8) and $m = 2$ gives

$$q_t + 6|q|^2 q_x + q_{xxx} = 0 \tag{2.11}$$

which is the modified KdV with complex field.

2.2. Recursion operators for KdV, SG and MKdV equations

First some notation: $\phi(x, t)$ will always represent a solution of the SG equation

$$\phi_{xt} = \sin\phi, \tag{2.12}$$

$v(x, t)$ a solution of the MKdV equation

$$v_t + \tfrac{3}{2}v^2 v_x + v_{xxx} = 0, \tag{2.13}$$

and finally $u(x, t)$ a solution of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.14}$$

An IT about $\phi(x, t)$ is represented by $y_{(\mathrm{s})}(x, t)$, that about $v(x, t)$ by $y_{(\mathrm{m})}(x, t)$ and that about $u(x, t)$ by $y_{(\mathrm{k})}(x, t)$, the subscripts standing for SG, MKdV and KdV respectively. Superscripts are used, for example, $y_{(\mathrm{s})}^{(n)}(x, t)$ to distinguish the IT obtained recursively through the recursion operator. Thus

$$y_{(\mathrm{s})}^{(n+1)}(x, t) = T_{\mathrm{s}}(\phi)\{y_{(\mathrm{s})}^{(n)}(x, t)\} \tag{2.15}$$

where $T_{\mathrm{s}}(\phi)$ is the recursion operator for an IT about $\phi(x, t)$, a solution of (2.12).

Using (2.6), one gets

$$2F_{11}^{(n+2)} = -r\int^x h^{(n)}(x_1, t)\,\mathrm{d}x_1 - r_x \int^x g^{(n)}(x_1, t)\,\mathrm{d}x_1 - rg^{(n)} + \tfrac{1}{2}(F_{11}^{(n)})_{xx}, \tag{2.16a}$$

$$2F_{11}^{(n+2)}=q\int^x h^{(n)}(x_1,t)\,\mathrm{d}x_1-q_x\int^x g^{(n)}(x_1,t)\,\mathrm{d}x_1-qg^{(n)}+\tfrac{1}{2}(F_{22}^{(n)})_{xx},\tag{2.16b}$$

where

$$h^{(n)}=q(F_{11}^{(n)})_x-r(F_{22}^{(n)})_x\tag{2.16c}$$

and $g^{(n)}$ is given by (2.6*c*).

KdV equation. With $r(x,t)=-1$, $q(x,t)=u(x,t)$, $F_{11}^{(n+2)}=y^{(n+1)}$ and $F_{11}^{(n)}=y^{(n)}$, equation (2.16*a*) becomes

$$T_{\mathrm{k}}^{\dagger}(u)\{y^{(n)}\}\equiv y^{(n+1)}=\frac{1}{4}\frac{\partial^2 y^{(n)}}{\partial x^2}+uy^{(n)}-\tfrac{1}{2}\int^x u_{x_1}y^{(n)}\,\mathrm{d}x_1.\tag{2.17}$$

$T_{\mathrm{k}}^{\dagger}(u)$ defined by (2.17) is the recursion operator for the IT about solutions of

$$\omega_t+3(\omega_x)^2+\omega_{xxx}=0,\tag{2.18}$$

which is obtained from (2.14) with $u(x,t)=\omega_x(x,t)$. The adjoint $T_{\mathrm{k}}(u)$ of $T_{\mathrm{k}}^{\dagger}(u)$,

$$T_{\mathrm{k}}(u)=\tfrac{1}{4}\,\partial^2/\partial x^2+u+\tfrac{1}{2}u_x\int^x \mathrm{d}x_1,\tag{2.19}$$

is the recursion operator for the IT about a solution $u(x,t)$ of (2.14). The hierarchy of NLEE obtained from

$$u_t+[T_{\mathrm{k}}(u)]^n\{u_x\}=0,\tag{2.20}$$

is well known (Lax 1968). That another hierarchy can be obtained using the inverse of $T_{\mathrm{k}}(u)$ will be shown in § 4.

Sine-Gordon. With $\phi_x=2q=-2r$, $F_{11}^{(n+2)}+F_{22}^{(n+2)}=y_{(\mathrm{s})}^{(n+1)}$ and $F_{11}^{(n)}+F_{22}^{(n)}=y_{(\mathrm{s})}^{(n)}$, one gets from (2.16*a*, *b*)

$$T_{\mathrm{s}}(\phi)\{y_{(\mathrm{s})}^{(n)}\}\equiv y_{(\mathrm{s})}^{(n+1)}=\tfrac{1}{4}\left((y_{(\mathrm{s})}^{(n)})_{xx}+\phi_x\int^x \phi_{x_1}(y_{(\mathrm{s})}^{(n)})_{x_1}\,\mathrm{d}x_1\right)\tag{2.21}$$

and

$$T_{\mathrm{s}}(\phi)=\tfrac{1}{4}\left(\partial^2/\partial x^2+\phi_x\int \mathrm{d}x_1\,\phi_{x_1}\,\partial/\partial x_1\right)\tag{2.22}$$

is the recursion operator for the IT about a solution $\phi(x,t)$ of (2.12).

Modified KdV. One similarly finds the recursion operator $T_{\mathrm{m}}(v)$ for the IT about a solution $v(x,t)$ of (2.13) to be

$$T_{\mathrm{m}}(v)=\tfrac{1}{4}\left(\partial^2/\partial x^2+v^2+v_x\int^x \mathrm{d}x_1\,v(x_1,t)\right).\tag{2.23}$$

Equation (2.23) can also be derived from (2.22) by noting that

$$v=\phi_x,\qquad y_{(\mathrm{m})}^{(n)}(x,t)=(y_{(\mathrm{s})}^{(n)}(x,t))_x.\tag{2.24}$$

If one considers the hierarchy of NLEE obtained from

$$\phi_t+[4T_{\mathrm{s}}(\phi)]^n\{\phi_x\}=0\tag{2.25}$$

one gets for $n = 1$

$$\phi_t + \tfrac{1}{2}(\phi_x)^3 + \phi_{xxx} = 0. \tag{2.26}$$

This is the MKdV equation (2.13) with $v = \phi_x$. One does not get the SG equation or its related hierarchy. We show in the next section that the inverse of $T_s(\phi)$ will generate the SG hierarchy, and our results agree with those obtained by Sasaki and Bullough (1981).

3. Inverse of $T_s(\phi)$

From (2.21) we have

$$y_{(s)}^{(n)}(x,t) = \int^x \mathrm{d}x_1 \left(4\int^{x_1} y_{(s)}^{(n+1)}(x_2,t)\,\mathrm{d}x_2 - \int^{x_1} \mathrm{d}x_2\, \phi_{x_2} \int^{x_2} \mathrm{d}x_3 (y_{(s)}^{(n)}(x_3,t))_{x_3}\phi_{x_3}\right) \tag{3.1}$$

which can be iterated to give

$$y_{(s)}^{(n)}(x,t) = 4\int^x \mathrm{d}x_1 \left(\sum_{n=0}^{\infty} (-1)^n I_{2n}(x_1,t)\right) \tag{3.2}$$

where

$$I_0(x,t) = \int^x y_{(s)}^{(n+1)}(x_1,t)\,\mathrm{d}x_1$$

and

$$I_{2n}(x,t) = \int^x \mathrm{d}x_1\, \phi_{x_1} \int^{x_1} \mathrm{d}x_2\, \phi_{x_2} \ldots \int^{x_{2n-1}} \mathrm{d}x_{2n}\, \phi_{x_{2n}} \int^{x_{2n}} y_{(s)}^{(n+1)}(x_{2n+1},t)\,\mathrm{d}x_{2n+1}. \tag{3.3}$$

ϕ_{x_i} are partial derivatives with respect to x_i. Integrating (3.3) by parts, we get

$$I_{2n} = \phi I_{2n-1} - \int^x \phi\phi_{x_1} I_{2n-2}(x_1,t)\,\mathrm{d}x_1. \tag{3.4}$$

Using (3.4), one can obtain the sum on the RHS of (3.2). The final result is

$$\begin{aligned} y_{(s)}^{(n)}(x,t) = 4\bigg(&\int^x \mathrm{d}x_1 \sin\phi(x_1,t) \int^{x_1} \mathrm{d}x_2 \sin\phi(x_2,t) \\ &+ \int^x \mathrm{d}x_1 \cos\phi(x_1,t)\int^{x_1} \mathrm{d}x_2 \cos\phi(x_2,t)\bigg)\{y_{(s)}^{(n+1)}(x_2,t)\}. \end{aligned} \tag{3.5}$$

Thus $[T_s(\phi)]^{-1}$, the inverse of $T_s(\phi)$, is

$$\begin{aligned} [T_s(\phi)]^{-1} = 4\bigg(&\int^x \mathrm{d}x_1 \sin\phi(x_1,t) \int^{x_1} \mathrm{d}x_2 \sin\phi(x_2,t) \\ &+ \int^x \mathrm{d}x_1 \cos\phi(x_1,t)\int^{x_1} \mathrm{d}x_2 \cos\phi(x_2,t)\bigg). \end{aligned} \tag{3.6}$$

We have verified that if $y_{(s)}(x,t)$ is an IT about $\phi(x,t)$ then so is $[T_s(\phi)]^{-1}\{y_{(s)}(x,t)\}$. Therefore $[T_s(\phi)]^{-1}$ is also a recursion operator for the IT. Since $\int^x \sin\phi(x_1,t)\,\mathrm{d}x_1$ is an IT about $\phi(x, t)$, one gets an infinity of IT given by

$$y_{(s)}^{(-n)}(x, t) = [T_s(\phi)]^{-n}\left\{\int^x \sin\phi(x_1, t)\,dx_1\right\}. \tag{3.7a}$$

Further, $[T_s(\phi)]^{-1}$ generates the hierarchy of NLEE

$$\phi_t + [T_s(\phi)]^{-n}\left\{\int^x \sin\phi(x_1, t)\,dx_1\right\} = 0. \tag{3.7b}$$

These are precisely the hierarchy of equations obtained by Sasaki and Bullough (1981).

The inverse of $T_m(v)$ can be immediately obtained by using (2.24) and (3.6) to give

$$[T_m(v)]^{-1} = 4\left[\sin\left(\int^x v(z, t)\,dz\right)\int^x dx_1 \sin\left(\int^{x_1} v(z, t)\,dz\right)\right.$$
$$\left. + \cos\left(\int^x v(z, t)\,dz\right)\int^x dx_1 \cos\left(\int^{x_1} v(z, t)\,dz\right)\right]\int^{x_1} dx_2. \tag{3.8}$$

The hierarchy of NLEE generated by $[T_m(v)]^{-1}$ is closely related to that obtained in (3.7).

4. Inverse of $T_k(u)$

The solutions $v(x, t)$ of (2.13) and $u(x, t)$ of (2.14) are related by the Miura transformation,

$$u = \tfrac{1}{4}v^2 - \tfrac{1}{2}\mathrm{i}v_x. \tag{4.1}$$

As usual, (4.1) can be linearised by

$$v = -2\mathrm{i}\psi_x/\psi \tag{4.2}$$

so that

$$u = -\psi_{xx}/\psi. \tag{4.3}$$

From (4.1) the relation between $y_{(k)}(x, t)$ and $y_{(m)}(x, t)$ is

$$y_{(k)}(x, t) = \tfrac{1}{2}[v(x, t)y_{(m)}(x, t) - \mathrm{i}(y_{(m)}(x, t))_x]. \tag{4.4}$$

A particular integral of (4.4) is

$$y_{(m)}(x, t) = 2\mathrm{i}\exp\left(-\mathrm{i}\int^x v\,dz\right)\int^x \exp\left(\mathrm{i}\int^{x_1} v\,dz\right)y_{(k)}(x_1, t)\,dx_1. \tag{4.5}$$

From (4.2),

$$\begin{aligned} &\exp -\mathrm{i}\int^x v\,dz = 1/\psi^2, \qquad \exp\mathrm{i}\int^x v\,dz = \psi^2, \\ &\sin\int^x v\,dz = (1/2\mathrm{i})(\psi^2 - 1/\psi^2), \qquad \cos\int^x v\,dz = \tfrac{1}{2}(\psi^2 + 1/\psi^2). \end{aligned} \tag{4.6}$$

From (3.8) one has

$$y_{(m)}^{(n-1)}(x,t)=[T_m(v)]^{-1}(y_{(m)}^{(n)}(x,t))$$
$$=4\left[\sin\left(\int^x v\,dz\right)\int^x dx_1\sin\left(\int^{x_1} v\,dz\right)\right.$$
$$\left.+\cos\left(\int^x v\,dz\right)\int^x dx_1\cos\left(\int^{x_1} v\,dz\right)\right]\left(\int^{x_1} dx_2\, y_{(m)}^{(n)}(x_2,t)\right). \tag{4.7}$$

Using (4.5), (4.6) and (4.7),

$$y_{(k)}^{(n-1)}(x,t)=(2\psi_x/\psi+\partial/\partial x)\left(-(\psi^2-1/\psi^2)\int^x dx_1\,(\psi^2-1/\psi^2)\right.$$
$$\left.+(\psi^2+1/\psi^2)\int^x dx_1\,(\psi^2+1/\psi^2)\right)\left\{\int^{x_1} dx_2\,(1/\psi)\int^{x_2} dx_3\,\psi^2 y_{(k)}^{(n)}\right\} \tag{4.8}$$

or finally

$$\tfrac{1}{4}[T_k(u)]^{-1}=\int^x dx_1\,(1/\psi^2)\int^{x_1} dx_2\,\psi^2$$
$$+2\psi\psi_x\int^x dx_1\,(1/\psi^2)\int^{x_1} dx_2\,(1/\psi^2)\int^{x_2} dx_3\,\psi^2 \tag{4.9}$$

where $\psi(x,t)$ is related to $u(x,t)$ by (4.3).

The IT $y_{(k)}(x,t)$ corresponding to the IT $y_{(m)}(x,t)=\sin(\int^x v\,dz)$ is, from (4.4) and (4.6),

$$y_{(k)}(x,t)=\psi\psi_x. \tag{4.10}$$

Constant factors multiplying IT can be ignored. The infinity of IT generated by $[T_k(u)]^{-1}$ is

$$[T_k(u)]^{-n}\{\psi\psi_x\} \tag{4.11}$$

which is distinct from those obtained by Wadati (1978), and the hierarchy of NLEE is

$$u_t+[T_k(u)]^{-n}\{\psi\psi_x\}=0$$

with

$$u=-\psi_{xx}/\psi. \tag{4.12}$$

$T_k(u)$ and $[T_k(u)]^{-1}$ are singular because, as easily verified,

$$T_k(u)\{\psi\psi_x\}=0 \tag{4.13a}$$

and

$$[T_k(u)]^{-1}\{u_x\}=0. \tag{4.13b}$$

Equation (4.13a) can also be understood in the following way. If Ψ satisfies the Schrödinger equation,

$$\Psi_{xx}+(u-\lambda)\Psi=0, \tag{4.14}$$

then $(\Psi^2)_x$ is an eigenfunction of $T_k(u)$ with eigenvalue λ. ψ as defined in (4.12) is (4.14) with $\lambda=0$ and (4.13a) follows.

It should be remarked that (4.12) can be written in an alternative way. For example, with $n = 0$ in (4.12) and operating with $T_k(u)$, we get

$$T_k(u)\{u_t\} = 0. \tag{4.15}$$

Such equations have been derived by Calogero and Degasperis (1976b). However, the resulting equations are not in the form of evolution equations. To put them in this form would be equivalent to finding $[T_k(u)]^{-1}$. Even so, it is necessary to put the equation in the form of coupled equations.

5. Conclusion and remarks

We have shown that a double infinity of IT can be obtained for the SG (hence MKdV) and the KdV equations through the corresponding recursion operators and their inverses. From these operators follow a double hierarchy of NLEE. The recursion operators have been obtained systematically from certain naturally occurring recursion relations in the generalised Wronskian techniques (Calogero and Degasperis 1976a). The recursion operators thus obtained (though not their inverses) have been derived by other authors (Ablowitz *et al* 1974, Newell and Flaschka 1975) in searching for operators whose eigenfunctions are related to the squares of eigenfunction of the associated spectral problem. However, it does not seem to have been recognised that these operators are recursion operators for IT. Further, the procedure used by these authors may not be applicable for $n \times n$ $(n > 2)$ spectral problems. We hope that the method outlined here would be more general.

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Note added in proof. After the manuscript was submitted we came to know of the work of Fokas and Fuchssteiner (1981). They have derived a general formula connecting the recursion operator of two NLEE in terms of the BT which takes the solution of one NLEE to a solution of the other. They have also noted that the inverse of $T_s(\phi)$ will lead to the sine-Gordon equation, though they have not found an explicit expression for the inverse as done here.

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